Chapter 0

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1 Exercises

0.1 Examine the following formal descriptions of sets so that you understand which members they contain. Write a short informal English description of each set.

(a) $\{1, 3, 5, 7, \ldots\}$
Set of odd, natural numbers.

(b) $\{x \ldots, -4, -2, 0, 2, 4 \ldots\}$
Set of even, integer numbers.

(c) $\{n \mid n = 2m \text{for some } m \in \mathbb{N}\}$
Set of even, natural numbers.

(d) $\{n \mid n = 2m \text{for some } m \in \mathbb{N}, \text{and} n = 3k \text{for some} k \in \mathbb{N}\}$
Set of even, natural numbers that are also multiples of 3.

(e) $\{w \mid w \text{is a string of } \textit{0s} \text{ and } \textit{1s} \text{ and } w \text{ equals the reverse of} w\}$
Set of strings of 0s and 1s that are palindroms

(f) $\{n \mid n \text{is an integer and} n = n + 1\}$
Set of integers that are equal to their successor which is *empty set*

0.2 Write formal descriptions of the following sets.

(a) The set containing the numbers 1, 10, and 100
$\{1, 10, 100\}$

(b) The set containing all integers that are greater than 5
$\{n \mid n > 5, n \in \mathbb{Z}\}$

(c) The set containing all natural numbers that are less than 5:
$\{n \mid n < 5, n \in \mathbb{N}\}$

(d) The set containing the string "aba":
$\{"aba"\}$

(e) The set containing the empty string:
$\{\epsilon\}$

(f) The set containing nothing at all:
$\emptyset$

0.3 Let A be the set $\{x, y, z\}$ and B be the set $\{x, y\}$

(a) Is A a subset of B? ($A \subseteq B$)
No, A is not a subset of B

(b) Is B a subset of A? ($B \subseteq A$)
Yes, B is a subset of A

(c) What is the union of A and B? ($A \cup B$)
$\{x, y, z\} = A$

(d) What is the intersection of A and B? ($A \cap B$)
$\{x, y\} = B$

(e) What is the cross product A and B? ($A \times B$)
$\{(x, x), (x, y), (y, x), (y, y), (z, x), (z, y)\}$

(f) What is the power set of B?
$\mathcal{P}(B) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$

0.4 If A has a elements and B has b elements, how many elements are in $A \times B$? Explain your answer.

The cross product of two sets A and B is the set of all possible ordered pairs of elements from A and B. If A has a elements and B has b elements, then the cross product $A \times B$ will have $a \cdot b$ elements.

0.5 If C is a set with c elements, how many elements are in the power set of C? Explain your answer.

The power set of a set C is the set of all possible subsets of C. If C has c elements, then the power set of C will have 2^c elements. Thats because every element can be present or absent in every subset, so there are 2^c possible combinations of elements in the power set of C.

0.6 Let X be the set $\{1, 2, 3, 4, 5\}$ and Y be the set $\{6, 7, 8, 9, 10\}$. The unary function $f : X \to Y$ and the binary function $g : X \times Y \to Y$ are described in the following tables.

n	f(n)
1	6
2	7
3	6
4	7
5	6

g(x,y)	6	7	8	9	10
1	10	10	10	10	10
2	7	8	9	10	6
3	7	7	8	8	9
4	9	8	7	6	10
5	6	6	6	6	6

(a) What is the value of f(2)?
$f(2) = 7$

(b) What are the range and domain of f?

- Range: $\{6, 7\}$
- Domain: $\{1, 2, 3, 4, 5\}$

(c) What is the value of $g(2, 10)$?
$g(2, 10) = 6$

(d) What are the range and domain of g?

- Range: $\{6, 7, 8, 9, 10\}$
- Domain: $\{1, 2, 3, 4, 5\}$

(e) What is the value of $g(4, f(4))$?
$g(4, f(4)) = 8$

0.7 For each part, give a relation that satisfies the condition.

(a) Reflexive [1] and symmetric [2] but not transitive.

(b) Reflexive and transitive [3] but not symmetric.

(c) Symmetric and transitive but not reflexive.

(a) Reflexive and symmetric but not transitive example is be coworker relation on the set of people (let's assume that everybody is a coworker with themselves)

(b) Reflexive and transitive but not symmetric example is greater than or equal to relation on the set of natural numbers

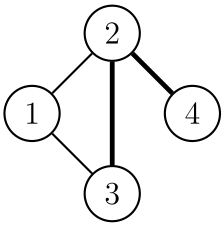
(c) Symmetric and transitive but not reflexive example is the relation "is a sibling of" on the set of people

[1] Reflexive relation R is when $\forall a \in A, (a, a) \in R$

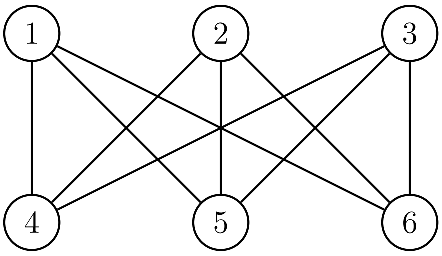
[2] Symmetric relation R is when $\forall a, b \in A, (a, b) \in R \implies (b, a) \in R$

[3] Transitive relation R is when $\forall a, b, c \in A, (a, b) \in R \land (b, c) \in R \implies (a, c) \in R$

0.8 Consider the undirected graph $G = (V, E)$ where V, the set of nodes, is $\{1, 2, 3, 4\}$ and E, the set of edges, is $\{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{2, 4\}, \{1, 4\}\}$. Draw the graph G. What are the degrees of each node? Indicate a path from node 3 to node 4 on your drawing of G.



0.9 Write a formal description of the following graph.



$G = (V, E)$ where

$V = \{1, 2, 3, 4, 5, 6\}$ and

$E = \{\{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{3, 6\}\}$

2 Problems

0.10 Find the error in the following proof that $2 = 1$.

Consider the equation $a = b$. Multiply both sides by a to obtain $a^2 = ab$. Subtract b^2 from both sides to get $a^2 - b^2 = ab - b^2$. Now factor each side, $(a + b)(a - b) = b(a - b)$, and divide each side by $(a - b)$ to get $a + b = b$. Finally, let a and b equal 1, which shows that $2 = 1$.

We cannot divide each side of equation $(a + b)(a - b) = b(a - b)$ by $(a-b)$ because $a = b$ implies $a-b = 0$ and division by zero is undefined.

0.11 Let $S(n) = 1 + 2 + \ldots + n$ be the sum of the first n natural numbers, and let $C(n) = 1^3 + 2^3 + \ldots + n^3$ be the sum of the first n cubes. Prove the following equalities by induction on n, to arrive at the curious conclusion that $C(n) = S^2(n)$ for every n.

1. $S(n) = \frac{1}{2}n(n + 1)$.

Basis $S(1) = 1 = \frac{1}{2}1(1+1)$.

Induction step

$S(k+1) = S(k) + (k+1) = \frac{1}{2}k(k+1) + (k+1) = \frac{1}{2}(k+1)(k+2)$.

2. $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n+1)^2$.

Basis

$C(1) = 1 = \frac{1}{4}1^2(1+1)^2$

Induction step

$C(k+1) = C(k) + (k+1)^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$.

The induction step is valid for both $S(n)$ and $C(n)$.

Therefore, $C(n) = S^2(n)$ for every n.

$S^2(n)$ is $(\frac{1}{2}n(n+1))^2 = \frac{1}{4}n^2(n+1)^2$ which is equal to $C(n)$.

0.12 Find the error in the following proof that all horses are the same color.

CLAIM: In any set of h horses, all horses are the same color.

PROOF: By induction on h.

Basis:

For $h = 1$. In any set containing just one horse, all horses clearly are the same color.

Induction step

For $k >= 1$, assume that the claim is true for $h = k$ and prove that it is true for $h = k + 1$. Take any set H of $k + 1$ horses. We show that all the horses in this set are the same color. Remove one horse from this set to obtain the set H_1 with just k horses. By the induction hypothesis, all the horses in H_1 are the same color. Now replace the removed horse and remove a different one to obtain the set H_2. By the same argument, all the horses in H_2 are the same color. Therefore, all the horses in H must be the same color, and the proof is complete.

The horse removed in the first step may have a different color than the horse removed in the second step. Therefore, the induction step is invalid.

0.13 Show that every graph with two or more nodes contains two nodes that have equal degrees.

A graph G is a pair of sets V and E where the elements of the non empty set V are called the vertices and the elements of a possibly empty set E, called the edges, are unordered pairs of vertices.[4]

...

Depending on the cardinality of the set V graphs may be finite or infinite. I will only consider finite graphs.

Narrowing the scope further: I shall only consider graphs with no loops and with no multiple edges - in what follows a pair of vertices may be connected with at most one edge.

A vertex v is incident with an edge e if v is a terminal point of e. As such, the degree of a vertex v is the number of edges incident with v. Intuitively - say, as a vertex I'm a city: the number of distinct roads that emanate from or lead into me is my degree.

Further, since our edge must connect exactly two distinct vertices, a and b, it follows that when we count the degree of a connected vertex, a or b, we "overcount its edge" (loosely speaking) - that edge contributes 1 toward the total degree of a and it also contributes 1 toward the total degree of b. Therefore, the sum of degrees d_i (some of which may be zero) of n vertices of a graph must be an even number (in e - the number of edges in a graph):

$$\sum_{i=1}^{n} = 1d_i = 2e$$

Now onto the theorem - proof by contradiction.

Observe that if the number of vertices in our graph is n then the largest degree of any vertex in such a graph must be strictly less then n:

$$maxd_i \leq n - 1$$

for any i. Why is that? By contradiction. Recall my roads leaving a city analogy - a road emanating from a given (and fixed) city has only and at most $n - 1$ other cities to go to since: no loops and no multiple roads to the same city are allowed.

Next, assume that, contrary to the conclusion, each and every vertex has a distinct degree. Therefore, our only choices (for these degrees) are:

[4]https://www.quora.com/How-would-we-prove-that-every-graph-with-at-least-two-vertices-has-two-vertices-of-the-same-degree

$D = 0, 1, 2, 3, \ldots, n-1$

But our graph can not have two vertices with degrees 0 and $n-1$ simultaneously.

Why is that?

By contradiction - assume that a graph has a vertex v with degree $n-1$. Then v must find exactly $n-1$ distinct cities on the other side. Therefore, in that case a vertex with degree 0 can not exist.

Therefore, our set of choices (3) splits into two: either:

$D = \{1, 2, 3, 4, \ldots, n-1\}$

or:

$D = \{0, 1, 2, 3, \ldots, n-2\}$

But in either case the cardinality of D is $n-1$:

$|D| = n-1$

Therefore, we have to distribute n vertices over $n-1$ degrees and by pigeonhole principle, since we have more vertices (pigeons) than degrees (pigeonholes):

at least two vertices must share the same degree

which amounts to a contradiction - our initial assumption that all the degrees are distinct was false. Therefore, if our graph has at least two vertices then at least two of them have the same degree.

0.14 Ramsey's theorem.

Let G be a graph. A clique in G is a subgraph in which every two nodes are connected by an edge. An anti-clique, also called an independent set, is a subgraph in which every two nodes are not connected by an edge. Show that every graph with n nodes contains either a clique or an anti-clique with at least $\frac{1}{2}log_2(n)$ nodes.

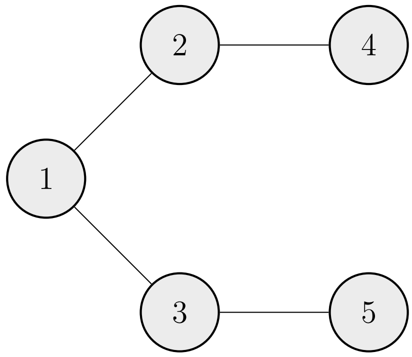
Practical example:

In a set of six individuals there is always a group of three people that either all know each other or where all are strangers to each other.

From book solutions:

Make space for two piles of nodes: C and AC. Then, starting with the entire graph, repeatedly add each remaining node x to C if its degree is greater than one half the number of remaining nodes and to AC otherwise, and discard all nodes to which x isn't (is) connected if it was added to C (AC). Continue until no nodes are left. At most half of the nodes are discarded at each of these steps, so at least $log_2 n$ steps will occur before the process terminates. Each step adds a node to one of the piles, so one of the piles ends up with at least $\frac{1}{2}log_2 n$ nodes. The A pile contains the nodes of a clique and the B pile contains the nodes of an anti-clique.

Visualise this procedure for graph:



step	vertex	remaining nodes	vertex degree	C	AC	discard
1	1	4	1	{}	{1}	2,3
2	4	1	0	{}	{1,4}	-
3	5	0	1	{}	{1,4,5}	-

At the end we get $AC = \{1, 4, 5\}$ which is an anti-clique with

$3 > \frac{1}{2}log_2 5 = 2$ nodes.

0.15 Use Theorem 0.25 to derive a formula for calculating the size of the monthly payment for a mortgage in terms of the principal P, the interest rate I, and the number of payments t. Assume that after t payments have been made, the loan amount is reduced to 0. Use the formula to calculate the dollar amount of each monthly payment for a 30-year mortgage with 360 monthly payments on an initial loan amount of $100,000 with a 5% annual interest rate.

Theorem 0.25

$\forall t \geq 0$

$P_t = PM^t - Y\frac{M^t-1}{M-1}$

$Y = PM^t(M-1)/(M^t-1)$

$P_t = 0$ $M = 1 + \frac{0.05}{12}$ $P = 100000$ $I = 0.05$ $t = 360$

$Y = 100000 \cdot (1 + \frac{0.05}{12})^{360} \cdot \frac{\frac{0.05}{12}}{(1+\frac{0.05}{12})^{360}-1} = 536.82$